EXPLICIT STILLMAN BOUNDS FOR ALL DEGREES

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Abstract. In 2016 Ananyan and Hochster proved Stillman's conjecture by showing the existence of a uniform upper bound for the projective dimension of all homogeneous ideals, in polynomial rings over a field, generated by n forms of degree at most d. Explicit values of the bounds for forms of degrees 5 and higher are not yet known.

The main result of this article is the construction of explicit such bounds, for all degrees d, which behave like power towers of height $\frac{1}{6}d^3 + \frac{11}{6}d - 4$. A key technical step is to establish a bound $D(k, d)$, which controls the number of generators of a minimal prime over an ideal of a regular sequence of k or fewer forms of degree d, and supplementing it into Ananyan and Hochster's proof in order to obtain a recurrence relation.

1. Introduction

Let K be a field, $R = K[x_1, \ldots, x_N]$ be a polynomial ring over K, and I be an ideal of R generated by n forms of degrees $d_1, \ldots, d_n$. We will denote the projective dimension of a module over R by $\mathrm{pd}_R(M)$. Stillman (see [21]) conjectured that the projective dimension of I can be bounded in terms of n and $d_1, \ldots, d_n$ but independent of N. We will refer to such bounds as Stillman bounds. Ananyan and Hochster were the first to give an affirmative answer to Stillman's conjecture in [2], where they showed the existence of Stillman bounds by proving the existence of small subalgebras and small subalgebra bounds ${}^{\eta}B$ (defined in [2, Theorem B] or Theorem 2.2). Stillman's conjecture was later reproved in [11] and [17], both using topological Noetherianity results from [12].

With the existence proven, the next question is to find explicit Stillman bounds. While many early and recent works [1] [3] [4] [5] [8] [16] [15] [18] [19] [20] have established Stillman bounds for degree 4 or less, the question for degree 5 and higher remains untouched. We give explicit Stillman bounds for all degrees by proving explicit small subalgebra bounds ${}^{\eta}B$ through a recurrence relation in Theorem 3.6. In particular the Stillman bound we obtain is estimated to be a power tower as follows:

Theorem 4.3. *If I is a homogeneous ideal in a polynomial ring R generated by n forms with maximum degree $d \geq 4$, then $pd_R(R/I)$ can be bounded by a power tower with base 7, height $\frac{1}{6}d^3 + \frac{11}{6}d - 4$, and top exponent $d + n + 3$.*

The main reason why Ananyan-Hochster's inductive proof does not produce explicit Stillman bounds is that the authors can only show the existence (see [2, §3]) of a bound called $D(k, d)$, which controls the number of generators of a minimal prime over an ideal of a regular sequence of k or fewer polynomials with degree d.

The crucial technical result of this article is the following lemma which establish an effective value for $D(k, d)$.

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Lemma 3.3. *Let K be an algebraically closed field, $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d. Then the minimal number of generators of P is bounded by*

$$D(k, d) = (2d)^{2^{{}^1\mathcal{B}(k,d)-1}}.$$

By inserting the above bound $D(k, d)$ into the double-layered induction of [2], we obtain a bound for ${}^\eta B$ and thus our Stillman bounds. Notice that in the above lemma, the assumption of the field being algebraically closed is needed only for existence of ${}^1\mathcal{B}(k, d)$ as in [2, Theorem B], and the proof still works if we drop this assumption and replace ${}^1\mathcal{B}(k, d)$ by $\mathrm{pd}_R(R/P)$ or N (see Theorem 3.5).

The paper is organized as follows. In Section 2, we first state the small subalgebra theorems of [2] after giving their necessary definitions, then set up notations needed for the next section. In Section 3, we construct the bound $D(k, d)$, discuss how we insert the bound $D(k, d)$ into Ananyan and Hochster's proof, and conclude the section with a theorem establishing our bounds for ${}^\eta B$ with a recurrence relation. In Section 3, we give an estimate of our bounds for ${}^\eta B$.

2. Notation

We first recall some definitions in [2, §1] which are needed for stating the theorems in [2]. Let $R = K[x_1, \dots, x_N]$ be a polynomial ring over a field K. Let V be a finite dimensional graded vector space of R, then we say V has *dimension sequence* $\delta = (\delta_1, \dots, \delta_d)$ if $V = V_1 \oplus \cdots \oplus V_d$ as a direct sum of its graded components with $\dim_K V_i = \delta_i$.

A function of several variables is called *ascending* if it is increasing in any one variable while the other variables are fixed.

A form $F \in R$ has a *k-collapse* if it can be written as a graded combination of k or fewer forms of strictly smaller positive degree. We say F has *strength* k if it has a $k+1$-collapse but no k-collapse. By convention we set the strength of a linear form to be $+\infty$.

A sequence of elements $G_1, \dots, G_s$ in a Noetherian ring R is a prime sequence (respectively, an *R_η-sequence*) if for $0 \leq i \leq s$, $R/(G_1, \dots, G_i)$ is a domain (respectively, satisfies the Serre condition R_η). When R is a polynomial ring, for any $\eta \geq 1$ an R_η-sequence is a prime sequence and hence a regular sequence.

Our goal is to give explicit bounds to the functions ${}^\eta A$ and ${}^\eta B$ for any degree, which are defined in Theorem A, Theorem B, and Corollary B of [2].

Theorem 2.1 (Ananyan-Hochster [2])**.** *There are ascending functions $A = (A_1, \dots, A_d)$ and, for every integer $\eta \geq 1$, ${}^\eta A = ({}^\eta A_1, \dots, {}^\eta A_d)$ from dimension sequences $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$ to $\mathbb{N}^d$ with the following property: For every algebraically closed field K and every positive integer N, if $R = K[x_1, \dots, x_N]$ is a polynomial ring, and V denotes a graded K-vector subspace of R of vector space dimension n with dimension sequence $(\delta_1, \dots, \delta_d)$, such that for $1 \leq i \leq d$, the strength of every nonzero element of V_i is at least $A_i(\delta)$ (respectively, ${}^\eta A_i(\delta)$), then every sequence of K-linearly independent forms in V is a regular sequence (respectively, is an R_η-sequence).*

Theorem 2.2 (Ananyan-Hochster [2])**.** *There is an ascending function B from dimension sequences $\delta = (\delta_1, \dots, \delta_d)$ to $\mathbb{Z}_+$ with the following property. If K is an algebraically closed field and V is a finite-dimensional $\mathbb{Z}_+$-graded K-vector subspace of a polynomial ring R over K with dimension sequence δ, then V (and, hence, the K-subalgebra of R generated by V) is contained in a K-subalgebra of R generated by a regular sequence $G_1, \dots, G_s$ of forms of degree* *at most d, where $s \le B(\delta)$. Moreover, for every $\eta \ge 1$ there is such a function ${}^{\eta}B$ with the additional property that every sequence consisting of linearly independent homogeneous linear combinations of the elements in $G_1, \dots, G_s$ is an R_η-sequence.*

The next corollary, as remarked in [2], follows immediately by taking ${}^{\eta}\mathcal{B}(n,d)$ to be the maximum of ${}^{\eta}B(\delta)$ over all dimension sequences with at most d entries and sum of entries at most n.

Corollary 2.3 (Ananyan-Hochster [2])**.** *There is an ascending function ${}^{\eta}\mathcal{B}(n,d)$, independent of K and N, such that for all polynomial rings $R = K[x_1, \dots, x_N]$ over an algebraically closed field K and all graded vector subspaces V of R of dimension at most n whose homogeneous elements have positive degree at most d, the elements of V are contained in a subring $K[G_1, \dots, G_B]$, where $B \le {}^{\eta}\mathcal{B}(n,d)$ and $G_1, \dots, G_B$ is an R_η-sequence of forms of degree at most d.*

We next introduce notations that are needed for the proof of Lemma 3.2 and Lemma 3.3. For a finitely generated graded R-module M, let $\beta_{ij}(M)$ be the graded Betti numbers of M and $\beta_i(M) = \sum_j \beta_{ij}(M)$ be the i-th Betti number of M. The Castelnuovo-Mumford regularity of M is defined as $\operatorname{reg}(M) = \max_{i,j}\{j - i : \beta_{ij}(M) \neq 0\}$.

Let J be a monomial ideal. We say J is *strongly stable* if for each monomial u of J, $x_i | u$ implies $x_j u / x_i \in I$ for each $j < i$. Let $G(J)$ be the set of minimal monomial generators of J and $D(J)$ be the largest degree of monomials in $G(J)$. If u is a monomial, let $m(u) := max\{i : x_i | u\}$. By the Eliahou-Kervaire resolution in [14], if J is strongly stable then $\beta_i(J) = \sum_{u \in G(J)} \binom{m(u)-1}{i}$.

Let I be a monomial ideal in $K[x_1, \dots, x_N]$ where K is an infinite field, we may assume I is generated by monic monomials, if K' is any other field then let $I_{K'}$ be the ideal generated by the image of these monomials in $K'[x_1, \dots, x_N]$. Let $\operatorname{gin}_{\text{rlex}}(I)$ be the generic initial ideal of I with respect to the degree reverse lexicographical order. The *zero-generic initial ideal* of I with respect to the degree reverse lexicographical order is defined to be

$$\operatorname{Gin}_0(I) := (\operatorname{gin}_{\text{rlex}}((\operatorname{gin}_{\text{rlex}}(I))_{\mathbb{Q}}))_K.$$

The zero-generic initial ideal is explored in more details in [7]. We need this notion in §3 to treat the positive characteristic cases. Notice that in characteristic 0, the zero-generic initial ideal is equal to the usual generic initial ideal.

3. Procedures to realize the bound via a recursive algorithm

We start this section by constructing a bound, denoted $D(k,d)$ in [2], for the number of generators of a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree d. We note that this bound is independent of the number of variables in the polynomial ring, which is necessary for our purpose. The following lemma, see [9, Theorem 27], tells us that the above minimal prime can be written as the ideal of the regular sequence colon by a form with bounded degree, which is key to the proof of Lemma 3.2.

Lemma 3.1 (Chardin [9])**.** *Let $P \subset K[x_1, \dots, x_N]$ be a minimal prime of an ideal generated by a homogeneous regular sequence $f_1, \dots, f_k$ of degrees $d_1, \dots, d_k$. There exists a form f of degree at most $d_1 + \cdots + d_k - k$ such that*

$$P = (f_1, \dots, f_k) : (f).$$

The next two lemmas justify the fact that we can choose $D(k,d)$ to be $(2d)^{2^{{}^1\mathcal{B}(k,d)-1}}$. Assuming the number of variables is known, we show in Lemma 3.2 how to bound the minimal number of generators of a particular kind of colon ideals as in Lemma 3.1. In pursuance of an optimal bound, we extensively apply results and proofs of [6] and [7]. For an alternate way to obtain a value for $D(k,d)$, we refer the reader to Remark 3.4, which could still be used to construct explicit but larger bounds for ${}^{\eta}B(\delta)$.

Lemma 3.2. *Let $I \subset K[x_1,\dots,x_{B+1}]$ be an ideal generated by a regular sequence of c forms of degree at most d, and f be a form of degree at most $cd-c$. Then the minimal number of generators of $I:(f)$ is bounded by*

$$\beta_0(I:(f)) \le B(d+1)(2d)\prod_{i=3}^{B}((d^2+2d-1)^{2^{i-3}}+1)+1 \le (2d)^{2^{B-1}},$$

where the last inequality holds if we further assume $B \ge 4$, or $B \ge 3$ and $d \ge 3$.

Proof. If $c=1$ then $\beta_0(I:(f))=1$, so assume $c \ge 2$. After a faithfully flat base change, we may assume K is infinite. Denote $R := K[x_1,\dots,x_{B+1}]$ and let $f_1,\dots,f_c$ be the regular sequence with $\deg(f_i) \le d$. Consider the exact sequence

$$0 \to R/(I:(f)) \xrightarrow{f} R/I \longrightarrow R/(I+(f)) \to 0.$$

Since $c \le \beta_0(I+(f)) \le c+1$, using the long exact sequence of $\mathrm{Tor}_i^R(-,K)$ induced from the above short exact sequence, we get $\beta_0(I:(f)) \le \beta_1(I+(f))+1$.

With the notations of Section 2, let $J := \mathrm{Gin}_0(I+(f))$. Denote $R_{[i]} := K[x_1,\dots,x_i]$. Let $(I+(f))_{\langle i\rangle}$ denote its image in $R/(l_{B+1},\dots,l_{i+1}) \cong R_{[i]}$ where $l_{B+1},\dots,l_{i+1}$ are general linear forms, let $J_{[i]}$ denote $J \cap R_{[i]}$. By [1, Proposition 2.2], J is strongly stable with $\beta_1(I+(f)) \le \beta_1(J)$. So by [14] and [6, Proposition 1.6], $\beta_1(J)=\sum_{u\in G(J)}(m(u)-1) \le B\cdot|G(J)| \le B\prod_{i=1}^{B}(D(J_{[i]})+1)$. Using [7, Theorem 2.20], we can get $D(J_{[i]}) \le \mathrm{reg}((I+(f))_{\langle i\rangle})$ for all i. Notice that $\mathrm{reg}((I+(f))_{\langle i\rangle}) \le id-i+1$ for all $i \le c$, because $m_{\langle i\rangle}^{id-i+1} \subseteq (I+(f))_{\langle i\rangle}$ where $m=(x_1,\dots,x_{B+1})$.

To bound $\mathrm{reg}((I+(f))_{\langle i\rangle})$ for $i \ge c+1$, we follow the proof of [6, Theorem 2.4 and Corollary 2.6]. Let $\lambda(M)$ denote the length of an Artinian module M. Using the same proof of [6, Theorem 2.4], we can get

$$\begin{aligned}\mathrm{reg}((I+(f))_{\langle i\rangle}) &\le \max\{d, cd-c, \mathrm{reg}((I+(f))_{\langle i-1\rangle})\}\\ &\qquad + \lambda\left(\frac{R_{[c]}}{(I+(f))_{\langle c\rangle}}\right)\prod_{j=c+2}^{i}\mathrm{reg}((I+(f))_{\langle j-1\rangle})\\ &\le \max\{d, cd-c, \mathrm{reg}((I+(f))_{\langle i-1\rangle})\} + d^c\prod_{j=c+1}^{i-1}\mathrm{reg}((I+(f))_{\langle j\rangle}).\end{aligned} \tag{3.1}$$

The last inequality holds since $R_{[c]}/(I+(f))_{\langle c\rangle}$ is a quotient ring of $R_{[c]}/(g_1,\dots,g_c)$, where $g_1,\dots,g_c$ is the image of $f_1,\dots,f_c$ in $R_{[c]}$ and is a regular sequence with $\deg(g_i) \le d$.

Now we use (3.1) recursively to bound $\mathrm{reg}((I+(f))_{\langle i\rangle})$ for $i \ge c+1$. Set $B_0 := cd-c+1$, recall that B_0 bounds $\mathrm{reg}((I+(f))_{\langle c\rangle})$. Apply 3.1 to $(I+(f))_{\langle c+1\rangle}$ to get $\mathrm{reg}((I+(f))_{\langle c+1\rangle}) \le cd-c+1+d^c =: B_1$. For $j \ge 2$, we set $B_j := B_{j-1}+d^c\prod_{k=1}^{j-1}B_k \le (B_{j-1})^2 \le (B_1)^{2^{j-1}}$. Hence for all $i \ge c+1$, $\mathrm{reg}((I+(f))_{\langle i\rangle}) \le B_{i-c} \le (cd-c+1+d^c)^{2^{i-c-1}} \le (d^2+2d-1)^{2^{i-3}}$, where the last inequality holds since the second last bound is decreasing as a function of c and $c \ge 2$.

If $B \geq 4$, then $B \leq 2^{B-2}$. Combining all the previous inequalities, we get

$$\begin{aligned}\beta_0(I:(f)) &\leq B(d+1)(2d)\prod_{i=3}^{B}((d^2+2d-1)^{2^{i-3}}+1)+1\\ &\leq 2^{B-2}(d+1)(2d)\prod_{i=3}^{B}(d^2+2d)^{2^{i-3}} \leq (2d)(2d)\prod_{i=3}^{B}(2d^2+4d)^{2^{i-3}}\\ &\leq (2d)^2\prod_{i=3}^{B}(2d)^{2^{i-2}} = (2d)^{2^{B-1}}.\end{aligned}$$

If $B = 3$ and $d \geq 3$, one easily checks that $\beta_0(I:(f)) \leq 3(d+1)(2d)(d^2+2d)+1 \leq (2d)^4$. $\square$

Lemma 3.3 constructs our value $(2d)^{2^{{}^1\mathcal{B}(k,d)-1}}$ for the bound $D(k,d)$ by passing to a polynomial subring with at most ${}^1\mathcal{B}(k,d)+1$ many variables first, using Corollary 2.3, then combining the result of Lemma 3.1 and Lemma 3.2.

Lemma 3.3. *Let K be an algebraically closed field, $P \subset K[x_1,\dots,x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d. Then the minimal number of generators of P is bounded by*

$$\beta_0(P) \leq (2d)^{2^{{}^1\mathcal{B}(k,d)-1}}.$$

Proof. Let $f_1,\dots,f_c$ be the regular sequence with $c \leq k$ and $\deg(f_i) \leq d$, let I be the ideal it generates. By Corollary 2.3, there exists a prime sequence $G_1,\dots,G_s$ with $s \leq {}^1\mathcal{B}(k,d)$ such that $f_1,\dots,f_c \in K[G_1,\dots,G_s]$. Denote $R = K[x_1,\dots,x_N]$ and $S = K[G_1,\dots,G_s]$. Then $\operatorname{pd}_R(R/I) = c \leq s$ and $\operatorname{pd}_S(S/P\cap S) \leq s$. Notice that R is a free and thus faithfully flat module over S since we can extend $G_1,\dots,G_s$ to a maximal regular sequence $G_1,\dots,G_N$ in R to get free extensions $K[G_1,\dots,G_s] \hookrightarrow K[G_1,\dots,G_N]$ and $K[G_1,\dots,G_N] \hookrightarrow K[x_1,\dots,x_N]$. Consequently we get $\operatorname{pd}_R(R/P) \leq s$ once we have shown $P = (P\cap S)R$. By faithfully flatness $f_1,\dots,f_c \in P\cap S$ remains a regular sequence in S and so $c = \operatorname{ht} P \geq \operatorname{ht}(P\cap S)R = \operatorname{ht} P\cap S \geq c$. Now by [2, Corollary 2.9], $(P\cap S)R$ is a prime ideal, therefore $P = (P\cap S)R$.

By 3.1, there exists a form $f \in R$ of degree at most $cd - c$ such that $P = I:(f)$. Consider the exact sequence

$$0 \to R/P \xrightarrow{f} R/I \longrightarrow R/(I+(f)) \to 0. \tag{3.2}$$

It follows that $\operatorname{pd}_R(R/(I+(f))) \leq s+1$. Then $\operatorname{depth} R/(I+(f)) \geq N-(s+1)$ by the Auslander-Buchsbaum formula. Let $l_{s+2},\dots,l_N \in R$ be a sequence of linear forms regular on R/P, $R/(I+(f))$, and $R/(f_1,\dots,f_c)$. Fix a graded isomorphism from $R/(l_{s+2},\dots,l_N)$ to $K[x_1,\dots,x_{s+1}]$, let $\overline{R}$ denote $K[x_1,\dots,x_{s+1}]$ and "—" denote the image of polynomials or ideals of R in $\overline{R}$. Notice that $\beta_0(P) = \beta_0(\overline{P})$. Since $l_{s+2},\dots,l_N$ is a regular sequence on $R/(I+(f))$, the short exact sequence in (3.2) remains exact after tensoring with $\overline{R}$. It follows that $\overline{P} = \overline{I}:(\overline{f})$. Notice that $\overline{f_1},\dots,\overline{f_c}$ is a regular sequence in $\overline{R}$, so we can apply Lemma 3.2 to get $\beta_0(\overline{P}) \leq (2d)^{2^{{}^1\mathcal{B}(k,d)-1}}$ for ${}^1\mathcal{B}(k,d) \geq 4$, or ${}^1\mathcal{B}(k,d) = 3$ and $d \geq 3$.

If ${}^1\mathcal{B}(k,d) = 2$, it is clear that $\beta_0(P) = \beta_0(P\cap S) \leq 2$. If ${}^1\mathcal{B}(k,d) = 3$ and $d = 2$, then P is a minimal prime of an ideal generated by two quadrics. Let $e(R/P)$ denote the Hilbert-Samuel multiplicity of R/P with respect ot the maximal ideal $(x_1,\dots,x_N)$, we have $e(R/P) \leq 4$. If either P contains a linear form or $e(R/P) = 4$, then P is a complete intersection and so

$\beta_0(P) = 2$. Otherwise P contains no linear forms and $e(R/P) = 3 = 1+\mathrm{ht}(P)$, so we can apply [13, Theorem 4.2] to see that R/P is Cohen-Macaulay and hence $\beta_0(P) \le e(R/P) = 3$. ☐

Remark 3.4. By using known results, there is a quick and simple way to obtain a worse upper bound than the one derived from Lemma 3.3. The outline is the following. First one uses the bound on the degrees of the generators of initial ideals given in [10, Corollary 3.6] and the proof of [10, Corollary 3.7] to bound the degrees of the generators of $\overline{P} = \overline{I} : (\overline{f})$. When $d \ge 2$, the bound one gets is $(d^c(cd-c))^{2^{s+1-c}} \le (d^2(2d-2))^{2^{{}^1\mathcal{B}(k,d)-1}} =: D$. Hence, by linear independency, the number of minimal homogeneous generators of $\overline{P}$ is at most $\binom{{}^1\mathcal{B}(k,d)+D}{D}$.

In the proof of Lemma 3.3, notice that we could replace ${}^1\mathcal{B}(k,d)$ by the projective dimension of R modulo the minimal prime P. The same proof will give us the bound:

Theorem 3.5. *Let K be an algebraically closed field, $P \subset K[x_1,\dots,x_N]$ be a minimal prime of an ideal generated by a regular sequence of k or fewer forms of degree at most d. Then the minimal number of generators of P is bounded by*

$$\beta_0(P) \le (2d)^{2^{pd(R/P)-1}} \le (2d)^{2^{N-1}}.$$

We conclude this section by presenting a recursive formula that computes ${}^\eta B(\delta)$, where $\delta = (\delta_1,\dots,\delta_d)$ is a dimension sequence (see Theorem 3.6). The theoretical proof of [2, §4], which contains an inductive argument on the degree d, can be easily made into an algorithm once we insert the bound obtained in Lemma 3.3.

Denote ${}^\eta A(i) = {}^3\mathcal{B}(D(\eta+1, i-1), i-1)+1$. Then by [2, Proposition 2.6 and Theorem A], we have ${}^\eta A_i(\delta) = {}^\eta A(i) + 3(\sum_{j=1}^d \delta_j - 1)$.

Section 4 of [2] explains how to obtain ${}^\eta B$ from ${}^\eta A$, which we will describe briefly as follows. Let V be any vector space with dimension sequence δ, if for all such V, the strength of every nonzero element of V_i is at least ${}^\eta A(\delta)$, then let ${}^\eta B(\delta) = \sum_{i=1}^d \delta_i$. Otherwise there exists a V and a degree i for which an element of V_i has an ${}^\eta A_i(\delta)$-collapse. In this case set ${}^\eta B(\delta) = \max_{\delta'}\{{}^\eta B(\delta')\}$, where δ' run through all dimension sequences derived from δ by keeping δ_j unchanged for $j > i$, decreasing δ_i by 1, and increasing the δ_j's for $j < i$ by a total of $2\cdot {}^\eta A_d(\delta)$. Notice that if $d = 1$, ${}^\eta B((\delta_1)) = \delta_1$ trivially satisfies 2.2.

By considering the worst case scenario in the algorithm described above, we may define ${}^\eta B$ recursively as

$$\begin{aligned} {}^\eta B(\delta_1,\dots,\delta_d) &= {}^\eta B(\delta_1,\dots,\delta_{d-2},\delta_{d-1} + 2\,{}^\eta A_d(\delta), \delta_d - 1) \\ &= {}^\eta B(\delta_1,\dots,\delta_{d-2}, a_i, \delta_d - i), \end{aligned}$$

where a_i satisfies the recurrence relation $a_0 = \delta_{d-1}$,

$$\begin{aligned} a_i &= a_{i-1} + 2({}^\eta A(d) + 3(a_{i-1} + \delta_1 + \cdots + \delta_{d-2} + \delta_d - (i-1) - 1)) \\ &= 7a_{i-1} + 2\,{}^\eta A(d) + 6(\delta_1 + \cdots + \delta_{d-2} + \delta_d) - 6i. \end{aligned}$$

Notice that $a_i = 7^i(\delta_1 + \cdots + \delta_d + \frac{1}{3}{}^\eta A(d) - \frac{7}{6}) + i - (\delta_1 + \cdots + \delta_{d-2} + \delta_d) - \frac{1}{3}{}^\eta A(d) + \frac{7}{6}$ is a solution of the recurrence relation. Our goal is to bring down all the degree d elements to degree $d-1$, and so we compute a_{δ_d} to get

$$\begin{aligned} &{}^\eta B(\delta_1,\dots,\delta_d) = {}^\eta B(\delta_1,\dots,\delta_{d-2}, a_{\delta_d}, 0) = \\ &\qquad {}^\eta B(\delta_1,\dots,\delta_{d-2}, 7^{\delta_d}(\delta_1 + \cdots + \delta_d + \frac{1}{3}{}^\eta A(d) - \frac{7}{6}) - (\delta_1 + \cdots + \delta_{d-2}) - \frac{1}{3}{}^\eta A(d) + \frac{7}{6}). \end{aligned}$$

With our recursive definition we have ${}^{\eta}\mathcal{B}(n,d) = {}^{\eta}B(\delta)$ where $\delta = (0, \cdots, 0, n)$ has d entries.

The algorithm finishes by iterating the above process until all forms are in degree 1. We summarize the above discussion into the following theorem.

Theorem 3.6. *Under the setting of Theorem 2.2, we have the bound ${}^{\eta}B(\delta) = b_{d-1}$ where b_{d-1} satisfies the following recurrence relation:*

$$b_i = 7^{b_{i-1}}\left(\frac{1}{3}{}^{3}\mathcal{B}(D(\eta+1, d-i), d-i) + b_{i-1} + \delta_1 + \cdots + \delta_{d-i} - \frac{5}{6}\right) - \frac{1}{3}{}^{3}\mathcal{B}(D(\eta+1, d-i), d-i) - (\delta_1 + \cdots + \delta_{d-i-1}) + \frac{5}{6}.$$

with $b_0 = \delta_d$, $D(\eta+1, d-i) = [2(d-i)]^{2^{{}^{1}\mathcal{B}(\eta+1,d-i)-1}}$ if $d-i > 1$, $D(\eta+1, 1) = \eta+1$, and ${}^{\eta}B((\delta_1)) = \delta_1$.

4. Estimate of the small subalgebra bound

In this section, we give an estimate for the bound ${}^{\eta}\mathcal{B}(n,d)$ in Corollary 2.3. Notice that this also estimates the bound ${}^{\eta}B(\delta)$ in Theorem 2.2 since if δ is a dimension sequence with d entries whose sum equals to n, then ${}^{\eta}B(\delta) \le {}^{\eta}\mathcal{B}(n,d)$.

Unfortunately a formula that represents the actual value of the recursion described in Theorem 3.6 will be a very complicated, hard to read formula that behaves like a power tower. We decided, for the sake of clarity, to push all the exponents up in order to obtain a reasonable upper bound written as a power tower with linear top exponent depending on d, n, η. We focus not on the optimal value of the top exponent, but on the height of power tower.

We denote a power tower with base a, height n, and top exponent x by

Notation 4.1. *Let $exp_a^n(x) = a^{a^{\cdot^{\cdot^{a^x}}}}$ with n a's.*

The result of our estimate, whose details are at the end of the section, is as follows.

Proposition 4.2. *Assume $d \ge 4$, then ${}^{\eta}\mathcal{B}(n,d)$ defined in Corollary 2.3 can be bounded by*

$$ {}^{\eta}\mathcal{B}(n,d) \le exp_7^{\frac{1}{6}d^3+\frac{11}{6}d-4}(d+n+\eta+2).$$

In particular when $\eta = 1$, ${}^{1}\mathcal{B}(n,d)$ gives a bound for the projective dimension.

Theorem 4.3. *If I is a homogeneous ideal in a polynomial ring R generated by n forms with maximum degree $d \ge 4$, then $pd(R/I)$ can be bounded by a power tower with base 7, height $\frac{1}{6}d^3 + \frac{11}{6}d - 4$, and top exponent $d+n+3$.*

For the rest of this section, we compute the estimate in 4.2. During the computations, we will push up the exponents repeatedly, by which we mean:

$$\text{If } a \ge 2, x, y, z \ge 0, \text{ and } x+y \le a^{y'}, \text{ then } a^{ya^z} + x \le a^{ya^z+x} \le a^{(x+y)a^z} \le a^{a^{y'}a^z} = a^{a^{y'+z}}.$$

For the computations below, we work with the recurrence in Theorem 3.6. Recall that ${}^{\eta}\mathcal{B}(n,d) = {}^{\eta}B(\delta)$ where $\delta = (0, \cdots, 0, n)$ has d entries, and ${}^{\eta}\mathcal{B}(n,d) = b_{d-1}$ with the b_i's defined in Theorem 3.6. We assume $d \ge 4$.

First we simplify the b_i's in Theorem 3.6.

$$\begin{aligned} b_i &= 7^{b_{i-1}}\left(\frac{1}{3}\,{}^3\mathcal{B}(D(\eta+1,d-i),d-i)+b_{i-1}-\frac{5}{6}\right)-\frac{1}{3}\,{}^3\mathcal{B}(D(\eta+1,d-i),d-i)+\frac{5}{6} \\ &\le 7^{b_{i-1}}(\frac{1}{3}\,{}^3\mathcal{B}(D(\eta+1,d-i),d-i)+b_{i-1}) \\ &\le 7^{2b_{i-1}}({}^3\mathcal{B}(D(\eta+1,d-i),d-i)). \end{aligned} \tag{4.1}$$

We push up the exponents appearing in ${}^\eta\mathcal{B}(n,d)=b_{d-1}$.

Notice that $2\,{}^3\mathcal{B}(D(\eta+1,d-i),d-i)\le 7^{{}^3\mathcal{B}(D(\eta+1,d-i),d-i)}$ and ${}^3\mathcal{B}(D(\eta+1,d-1),d-1)\ge {}^3\mathcal{B}(D(\eta+1,d-i),d-i)$ for all $i\ge 1$.

$$\begin{aligned} {}^\eta\mathcal{B}(n,d)=b_{d-1} &\le {}^3\mathcal{B}(D(\eta+1,1),1)7^{2\,{}^3\mathcal{B}(D(\eta+1,2),2)7^{\cdot^{\cdot^{2\,{}^3\mathcal{B}(D(\eta+1,d-1),d-1)7^{2n}}}}} \\ &\le (\eta+1)\exp_7^{d-1}({}^3\mathcal{B}(D(\eta+1,2),2)+\cdots+{}^3\mathcal{B}(D(\eta+1,d-1),d-1)+2n) \\ &\le (\eta+1)\exp_7^{d-1}((d-2)\,{}^3\mathcal{B}(D(\eta+1,d-1),d-1)+2n). \end{aligned} \tag{4.2}$$

Apply (4.2) to ${}^3\mathcal{B}(D(\eta+1,d-1),d-1)$, we get:

$${}^3\mathcal{B}(D(\eta+1,d-1),d-1)\le 4\exp_7^{d-2}((d-3)\,{}^3\mathcal{B}(D(4,d-2),d-2)+2D(\eta+1,d-1)) \tag{4.3}$$

Apply (4.2) to ${}^3\mathcal{B}(D(4,d-2),d-2), {}^3\mathcal{B}(D(4,d-3),d-3),\dots,{}^3\mathcal{B}(D(4,2),2)$ and push up all the exponents, we get:

$$\begin{aligned} {}^3\mathcal{B}(D(4,d-2),d-2) &\le 4\exp_7^{d-3}((d-4)\,{}^3\mathcal{B}(D(4,d-3),d-3)+2D(4,d-2)) \\ &\le \exp_7^{d-3}((d-4)\,{}^3\mathcal{B}(D(4,d-3),d-3)+2D(4,d-2)+1) \\ &\le \exp_7^{\sum_{j=1}^{d-3}j}(\sum_{j=2}^{d-4}j+2(d-3)D(4,d-2)+d-3) \\ &= \exp_7^{\frac{d^2-5d+6}{2}}(2(d-3)D(4,d-2)+\frac{1}{2}d^2-\frac{5}{2}d+2). \end{aligned} \tag{4.4}$$

Now we combine the above inequalities. The first inequality below follows from (4.2), the second from (4.3), the third from $4(d-2)\le 7^{d-2}$, the fourth from (4.4) and $(d-3)+(d-2)\le 7^{\frac{1}{2}d-1}$.

$$\begin{aligned} {}^\eta\mathcal{B}(n,d) &\le (\eta+1)\exp_7^{d-1}((d-2)\,{}^3\mathcal{B}(D(\eta+1,d-1),d-1)+2n) \\ &\le (\eta+1)\exp_7^{d-1}((d-2)4\exp_7^{d-2}((d-3)\,{}^3\mathcal{B}(D(4,d-2),d-2)+2D(\eta+1,d-1)+n)) \\ &\le (\eta+1)\exp_7^{d-1+d-2}((d-3)\,{}^3\mathcal{B}(D(4,d-2),d-2)+2D(\eta+1,d-1)+n+(d-2)) \\ &\le (\eta+1)\exp_7^{2d-3+\frac{d^2-5d+6}{2}}(2(d-3)D(4,d-2)+\frac{1}{2}d^2-\frac{5}{2}d+2+D(\eta+1,d-1)+n+\frac{1}{2}d-1) \\ &= (\eta+1)\exp_7^{\frac{d^2-d}{2}}(2(d-3)D(4,d-2)+D(\eta+1,d-1)+n+\frac{1}{2}d^2-2d+1). \end{aligned} \tag{4.5}$$

In particular when $\eta = 1$:

$$\begin{aligned} {}^1\mathcal{B}(n,d) &\leq \exp_7^{\frac{d^2-d}{2}}(2(d-3)D(4,d-2)+D(2,d-1)+n+\frac{1}{2}d^2-2d+2) \\ &\leq \exp_7^{\frac{d^2-d}{2}}((2d-5)D(4,d-1)+n+\frac{1}{2}d^2-2d+2). \end{aligned}$$

We bound $D(\eta+1,d-1)$ as:

$$\begin{aligned} D(\eta+1,d-1) &= [2(d-1)]^{2^{{}^1\mathcal{B}(\eta+1,d-1)-1}} \\ &\leq [7^{d-1}]^{7^{\exp_7^{\frac{d^2-3d+2}{2}}((2d-7)D(4,d-2)+\eta+\frac{1}{2}d^2-3d+\frac{11}{2})}} \\ &\leq \exp_7^{\frac{d^2-3d+6}{2}}((2d-7)D(4,d-2)+\eta+\frac{1}{2}d^2-2d+\frac{9}{2}). \end{aligned} \tag{4.6}$$

To estimate $D(4,d-2)$, notice that for $3 \leq j \leq d-2$, we have:

$$\begin{aligned} D(4,j) &= [8j]^{2^{{}^1\mathcal{B}(4,j)-1}} \\ &\leq [7^j]^{7^{\exp_7^{\frac{j^2-j}{2}}((2j-5)D(4,j-1)+4+\frac{1}{2}j^2-2j+2)}} \\ &\leq \exp_7^{\frac{j^2-j+4}{2}}((2j-5)D(4,j-1)+\frac{1}{2}j^2-j+6). \end{aligned} \tag{4.7}$$

And when $j = 2$, we get the bound below. Notice that we can use the first line of (4.1) to bound ${}^1B(4,2) = 7^4(\frac{1}{3}\cdot 2+4-\frac{5}{6})-\frac{1}{3}\cdot 2+\frac{5}{6} \leq 4\cdot 7^4$.

$$D(4,2) = [4]^{2^{{}^1B(4,2)-1}} \leq [4]^{2^{4\cdot 7^4-1}} \leq \exp_7^3(5).$$

Apply (4.7) recursively, use the inequality $\sum_{j=3}^{d-2}(2j-5)+(\frac{1}{2}j^2-j+6) \leq \exp_7^2(\frac{1}{2}d-2)$, and push up the exponents to see:

$$D(4,d-2) \leq \exp_7^{\sum_{j=2}^{d-2}\frac{j^2-j+4}{2}}(\frac{1}{2}d-2+5) = \exp_7^{\frac{d^3-6d^2+23d-42}{6}}(\frac{1}{2}d+3). \tag{4.8}$$

We combine the previous inequalities to get our final bound below. Notice that the first inequality follows from (4.5), the second from (4.6), the fourth from (4.8) and the inequality $d^2-3d+\frac{7}{2}+(2d-6) \leq \exp_7^2(\frac{1}{2}d-1)$.

$$\begin{aligned} {}^\eta\mathcal{B}(n,d) &\leq \exp_7^{\frac{d^2-d}{2}}(2(d-3)D(4,d-2)+D(\eta+1,d-1)+n+\frac{1}{2}d^2-2d+2+\eta) \\ &\leq \exp_7^{d^2-2d+3}(d-3+(2d-6)D(4,d-2)+\eta+\frac{1}{2}d^2-2d+\frac{9}{2} \\ &\qquad +n+\frac{1}{2}d^2-2d+2+\eta) \\ &= \exp_7^{d^2-2d+3}((2d-6)D(4,d-2)+2\eta+n+d^2-3d+\frac{7}{2}) \\ &\leq \exp_7^{d^2-2d+3+\frac{d^3-6d^2+23d-42}{6}}(\frac{1}{2}d+3+\eta+n+\frac{1}{2}d-1) \\ &= \exp_7^{\frac{1}{6}d^3+\frac{11}{6}d-4}(d+2+n+\eta). \end{aligned}$$

In particular when $\eta = 1$, we get ${}^1\mathcal{B}(n,d) \leq \exp_7^{\frac{1}{6}d^3+\frac{11}{6}d-4}(d+n+3)$.

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